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SIMULTANEOUS INCREASE IN STEP - NUMBER AND DERIVATIVE ORDER AND ITS EFFECT ON ACCURACY OF MULTIDERIVATIVE MULTISTEP METHOD.

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Abstract

This paper discusses the effect of simultaneously increase in step - number (k) and order of derivative (*l*) on the accuracy of an implicit multiderivative method. The study varied k and *l* simultaneously from 1- 4 to produce some variants of the method. These variants were implemented by using them to solve two initial value problems of first order ordinary differential equations. A comparative study of the computed results was carried out which showed an improvement in accuracy as k and *l* increased from k=1; *l* =1 to k=2; *l* =2 but accuracy reduced from k=3; *l* =3 to k=4; *l* =4, suggesting that two step second derivative scheme (k=2; *l*=2) has optimal accuracy when k and *l* were increased simultaneously.

Keywords:

Step - number;
Implicit;
First Derivative;
Accuracy;
Ordinary Differential equation.

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1. Introduction

Differential equations form the language in which the basic laws of physical science are expressed and they constitute a large and very important aspect of today's mathematics. They occur in connection with the mathematical description of problems that are encountered in various branches of science whenever a relationship involving some continuously changing quantities and their rates of change is known or formed [1]. Differential equations are conventionally solved through analytical techniques which consist of expressing the solution in terms of some elementary functions, such as Bessel, trigonometry functions and lots of other functions, but complexity of natural processes leads to mathematical models which cannot be completely solved by analytical techniques, fundamental importance of numerical simulation is therefore gaining insight. [2]. Numerical methods which can give approximate solutions to an appreciable degree of accuracy therefore provide a powerful alternative tool for solving these differential equations [1] and [3].

There is need to work on the existing numerical methods in order to improve the accuracy, efficiency and effectiveness. Generally, every attempt to improve an existing numerical method usually results into developing another method which hopefully will be better [4].

According to [5], one of the ways of improving the accuracy of a linear one - step numerical method is by increasing the step number, for example, an improvement on the linear one - step methods like Euler and Runge – Kutta methods led to the development of the more accurate linear multistep methods such as; Adams Bashforth method, Adams Moulton method, Backward Differentiation Formula, and so on. This was confirmed by Famurewa and Olorunsola in [6].

According to [7], another way of improving accuracy and absolute stability of a linear multistep method is by using not only the function y and its derivative but also its higher derivatives. A multi - derivative method involves more analytical properties of the differential equation by involving more derivative properties of y of the method. [8] and [9]. For example, 1-stage third derivative multistep methods (TDMMs) that used first, second and third derivatives of the solution were developed by Ezzeddine and Hojjati in [7]. Also, a one -

step multiderivative method was developed by Famurewa and Olorunsola in [10], the developed method was tested to be more accurate and A - stable when compared with some existing linear multistep methods.
In this paper, attempt was made to investigate the pattern of accuracy of the schemes as the step number (k) and order of derivative (*l*) are increased together.

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2. Research Method

The local truncation error associated with the multiderivative method

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = \sum_{i=1}^{l} h^i \sum_{j=0}^{k} \beta_{ij} y^i{}_{n+j}, \quad \alpha_k = +1 \tag{1}$$

is

$$T_{n+k} = \sum_{j=0}^{k} \alpha_j y_{n+j} - \sum_{i=1}^{l} h^i \sum_{j=0}^{k} \beta_{ij} y^i{}_{n+j} \; ; \; \alpha_k = +1 \tag{2}$$

The development of the schemes adopted Taylor series expansion of the variables

$y^i{}_{n+j}$; $i = 0(1)l \; and \; j = 0(1)k$ given as;

$$y^i{}_{n+j} = \sum_{r=0}^{\infty} \frac{(jh)^r y^{r+i}}{r!}, \tag{3}$$

Accuracy of order P was imposed on the local truncation error T_{n+k} and the resulting equations were solved for parameters α_{js} and $\beta_{i,js}$ to generate the required schemes.

Derivation of the one - step first derivative scheme

Setting *k=1, l=1* in (3.1) gives

$$\alpha_0 y_n + \alpha_1 y_{n+1} = h\beta_{10} y^{(1)}{}_n + h\beta_{11} y^{(1)}{}_{n+1} \tag{4}$$

with local truncation error

$$T_{n+1} = \alpha_0 y_n + \alpha_1 y_{n+1} - h\beta_{10} y^{(1)}{}_n - h\beta_{11} y^{(1)}{}_{n+1} \tag{5}$$

adopting Taylor's series expansion of y_{n+1} and $y^{(1)}_{n+1}$ in (5) and combining terms in equal powers of h gives

$$T_{n+1} = (\alpha_0 + \alpha_1) y_n + (\alpha_1 - \beta_{10} - \beta_{11}) h y_n^{(1)} + \left(\frac{\alpha_1}{2} - \beta_{11}\right) h^2 y_n^{(11)} + \left(\frac{\alpha_1}{6} - \frac{\beta_{11}}{2}\right) h^3 y^{(111)}{}_n + 0(h^4) \tag{6}$$

that is;

$$T_{n+1} = C_0 y_n + C_1 h y_n^{(1)} + C_2 h^2 y^{(11)}{}_n + C_3 h^3 y^{(111)}{}_n + \ldots + 0(h^4)$$

where

$$C_0 = \alpha_0 + \alpha_1$$

$$C_1 = \alpha_1 - \beta_{10} - \beta_{11}$$

$$C_2 = \frac{\alpha_1}{2} - \beta_{11}$$

$$C_3 = \frac{\alpha_1}{6} - \tfrac{1}{2}\beta_{11} \tag{7}$$

imposing accuracy of order 2 on T_{n+1} to have
$C_0 = C_1 = C_2 = 0$ and $T_{n+1} = 0(h^3)$,
solving gives;
$\alpha_0 = -1$, $\beta_{10} = ½$, and $\beta_{11} = ½$.
putting these values into equation (4) gives the one – step, first derivative method :

$$y_{n+1} = y_n + \frac{h}{2}(y_{n+1}^{(1)} + y_n^{(1)}) \tag{8}$$

Derivation of the two - step second derivative scheme
Setting $k=2$, $l=2$ in (3.1) gives

$$\alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_2 y_{n+2} = h\left(\beta_{10} y_n^{(1)} + \beta_{11} y^{(1)}{}_{n+1} + \beta_{12} y^{(1)}{}_{n+2}\right) + h^2\left(\beta_{20} y^{(11)}{}_n + \beta_{21} y^{(11)}_{n+1} + \beta_{22} y^{(11)}_{n+2}\right) \tag{9}$$

with local truncation error

$$T_{n+2} = \alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_2 y_{n+2} - h\left(\beta_{10} y_n^{(1)} + \beta_{11} y^{(1)}{}_{n+1} + \beta_{12} y^{(1)}{}_{n+2}\right) - h^2\left(\beta_{20} y^{(11)}{}_n + \beta_{21} y^{(11)}_{n+1} + \beta_{22} y^{(11)}_{n+2}\right) \tag{10}$$

adopting Taylor's series expansion of y_{n+1}, y_{n+2}, $y_{n+1}^{(1)}$, $y_{n+2}^{(1)}$, $y_{n+1}^{(11)}$ and $y_{n+2}^{(11)}$ in (10) and combining terms in equal powers of h gives

$$T_{n+2} = C_0 y_n + C_1 h y_n^{(1)} + C_2 h^2 y^{(11)}{}_n + C_3 h^3 y^{(111)}{}_n + \ldots + 0(h^8) \tag{11}$$

where

$$C_0 = \alpha_0 + \alpha_1 + \alpha_2$$

$$C_1 = \alpha_1 + 2\,\alpha_2 - \beta_{10} - \beta_{11} - \beta_{12}$$

$$C_2 = \frac{\alpha_1}{2} + 2\,\alpha_2 - \beta_{11} - 2\beta_{12} - \beta_{20} - \beta_{22}$$

$$C_3 = \frac{\alpha_1}{3!} + \frac{4\alpha_2}{3} - \tfrac{1}{2}\beta_{11} - 2\beta_{12} - \beta_{21} - 2\beta_{22}$$

$$C_4 = \frac{\alpha_1}{4!} + \frac{2\alpha_2}{3} - \tfrac{1}{3!}\beta_{11} - \frac{4}{3}\beta_{12} - \tfrac{1}{2}\beta_{21} - 2\beta_{22} \tag{12}$$

$$C_5 = \frac{\alpha_1}{5!} + \frac{4\alpha_2}{15} - \tfrac{1}{4!}\beta_{11} - \frac{2}{3}\beta_{12} - \tfrac{1}{3!}\beta_{21} - \frac{4}{3}\beta_{22}$$

$$C_6 = \frac{\alpha_1}{6!} + \frac{4\alpha_2}{45} - \tfrac{1}{5!}\beta_{11} - \frac{4}{15}\beta_{12} - \tfrac{1}{4!}\beta_{21} - \frac{2}{3}\beta_{22}$$

$$C_7 = \frac{\alpha_1}{7!} + \frac{4\alpha_2}{315} - \tfrac{1}{6!}\beta_{11} - \frac{4}{45}\beta_{12} - \tfrac{1}{5!}\beta_{21} - \frac{4}{15}\beta_{22}$$

imposing accuracy of order 7 on T_{n+2} to have
$C_0 = C_1 = C_2 = \ldots = C_6 = C_7 = 0$ and $T_{n+2} = 0(h^8)$
solving gives;
$\alpha_0=-1$, $\alpha_1=-2$, $\beta_{10}=0.3750$, $\beta_{11}=0.0000$, $\beta_{12}=0.3750$, $\beta_{20}=0.0417$, $\beta_{21}=0.3333$ and $\beta_{22}=0.0417$
putting these values into equation (9) gives two-step second derivative method of the form;

$$y_{n+2} = 2y_{n+1} - y_n + h\left(0.38 y^{(1)}{}_{n+2} - 0.38\, y^{(1)}_n\right) - h^2\left(0.04 y^{(11)}{}_n - 0.33 y^{(11)}{}_{n+1} + 0.04\, y^{(11)}{}_{n+2}\right) \tag{13}$$

Derivation of the three - step third derivative scheme
Setting $k=3$, $l=3$ in (3.1) gives

$$\begin{gathered}\alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_2 y_{n+2} + \alpha_3 y_{n+3} = \\ h\left(\beta_{10} y_n^{(1)} + \beta_{11} y_{n+1}^{(1)} + \beta_{12} y_{n+2}^{(1)} + \beta_{13} y_{n+3}^{(1)}\right) \\ + h^2\left(\beta_{20} y_n^{(11)} + \beta_{21} y_{n+1}^{(11)} + \beta_{22} y_{n+2}^{(11)} + \beta_{23} y_{n+3}^{(11)}\right) \\ + h^3\left(\beta_{30} y_n^{(111)} + \beta_{31} y_{n+1}^{(111)} + \beta_{32} y_{n+2}^{(111)} + \beta_{33} y_{n+3}^{(111)}\right)\end{gathered} \tag{14}$$

with local truncation error

$$T_{n+3} = \alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_2 y_{n+2} - h(\beta_{10} y_n^{(1)} + \beta_{11} y_{n+1}^{(1)} + \beta_{12} y_{n+2}^{(1)} + \beta_{13} y_{n+3}^{(1)})$$
$$-h^2(\beta_{20} y_n^{(11)} + \beta_{21} y_{n+1}^{(11)} + \beta_{22} y_{n+2}^{(11)} + \beta_{23} y_{n+3}^{(11)}) - h^3(\beta_{30} y_n^{(111)} + \beta_{31} y_{n+1}^{(111)} + \beta_{32} y_{n+2}^{(111)} + \beta_{33} y_{n+3}^{(111)})$$

(15)

adopting Taylor's series expansion of $y_{n+1}, y_{n+1}^{(1)}, y_{n+1}^{(11)}, y_{n+1}^{(111)}, y_{n+2}, y_{n+2}^{(1)}, y_{n+2}^{(11)}, y_{n+2}^{(111)}$ $y_{n+3}, y_{n+3}^{(1)}, y_{n+3}^{(11)}$ and $y_{n+3}^{(111)}$ in (15) and combining terms in equal powers of h gives

$$T_{n+3} = C_0 y_n + C_1 h y_n^{(1)} + C_2 h^2 y_n^{(11)} + C_3 h^3 y_n^{(111)} + \ldots + 0(h^{14})$$

where

$$C_0 = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3$$
$$C_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 - \beta_{10} - \beta_{11} - \beta_{12} - \beta_{13}$$
$$C_2 = \frac{\alpha_1}{2} + 2\alpha_2 + \frac{9\alpha_3}{2} - \beta_{11} - 2\beta_{12} - 3\beta_{13} - \beta_{20} - \beta_{21} - \beta_{22} - \beta_{23}$$
$$C_3 = \frac{\alpha_1}{3!} + \frac{4\alpha_2}{3} + \frac{9\alpha_3}{2} - \frac{1}{2}\beta_{11} - 2\beta_{12} - \frac{9}{2}\beta_{13} - \beta_{21} - 2\beta_{22} - 3\beta_{23} - \beta_{30} - \beta_{31} - \beta_{32} - \beta_{33}$$
$$C_4 = \frac{\alpha_1}{4!} + \frac{2\alpha_2}{3} + \frac{27\alpha_3}{8} - \frac{1}{3!}\beta_{11} - \frac{4}{3}\beta_{12} - \frac{9}{2}\beta_{13} - \frac{1}{2}\beta_{21} - 2\beta_{22} - \frac{9}{2}\beta_{23} - \beta_{31} - 2\beta_{32} - 3\beta_{33}$$
$$C_5 = \frac{\alpha_1}{5!} + \frac{4\alpha_2}{15} + \frac{81\alpha_3}{40} - \frac{1}{4!}\beta_{11} - \frac{2}{3}\beta_{12} - \frac{27}{8}\beta_{13} - \frac{1}{3!}\beta_{21} - \frac{4}{3}\beta_{22} - \frac{9}{2}\beta_{23} - \frac{1}{2}\beta_{31} - 2\beta_{32} - \frac{9}{2}\beta_{33}$$
$$C_6 = \frac{\alpha_1}{6!} + \frac{4\alpha_2}{45} + \frac{81\alpha_3}{80} - \frac{1}{5!}\beta_{11} - \frac{4}{15}\beta_{12} - \frac{81}{40}\beta_{13} - \frac{1}{4!}\beta_{21} - \frac{2}{3}\beta_{22} - \frac{27}{8}\beta_{23} - \frac{1}{3!}\beta_{31} - \frac{4}{3}\beta_{32} - \frac{9}{2}\beta_{33}$$
$$C_7 = \frac{\alpha_1}{7!} + \frac{8\alpha_2}{315} + \frac{243\alpha_3}{560} - \frac{1}{6!}\beta_{11} - \frac{4}{45}\beta_{12} - \frac{81}{80}\beta_{13} - \frac{1}{5!}\beta_{21} - \frac{4}{15}\beta_{22} - \frac{81}{40}\beta_{23} - \frac{1}{4!}\beta_{31} - \frac{2}{3}\beta_{32} - \frac{27}{8}\beta_{33}$$
$$C_8 = \frac{\alpha_1}{8!} + \frac{2\alpha_2}{315} + \frac{729\alpha_3}{4480} - \frac{1}{7!}\beta_{11} - \frac{8}{315}\beta_{12} - \frac{243}{560}\beta_{13} - \frac{1}{6!}\beta_{21} - \frac{4}{45}\beta_{22} - \frac{81}{80}\beta_{23} - \frac{1}{5!}\beta_{31} - \frac{4}{15}\beta_{32} - \frac{81}{40}\beta_{33}$$

$$C_0 = \alpha_0+\alpha_1+\alpha_2+\alpha_3$$

$$C_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 - \beta_{10} - \beta_{11} - \beta_{12} - \beta_{13}$$

$$C_2 = \frac{\alpha_1}{2} + 2\alpha_2 + \frac{9\alpha_3}{2} - \beta_{11} - 2\beta_{12} - 3\beta_{13} - \beta_{20} - \beta_{21} - \beta_{22} - \beta_{23}$$

$$C_3 = \frac{\alpha_1}{3!} + \frac{4\alpha_2}{3} + \frac{9\alpha_3}{2} - \tfrac{1}{2}\beta_{11} - 2\beta_{12} - \frac{9}{2}\beta_{13} - \beta_{21} - 2\beta_{22} - 3\beta_{23} - \beta_{30} - \beta_{31} - \beta_{32} - \beta_{33}$$

$$C_4 = \frac{\alpha_1}{4!} + \frac{2\alpha_2}{3} + \frac{27\alpha_3}{8} - \tfrac{1}{3!}\beta_{11} - \frac{4}{3}\beta_{12} - \frac{9}{2}\beta_{13} - \frac{1}{2}\beta_{21} - 2\beta_{22} - \frac{9}{2}\beta_{23} - \beta_{31} - 2\beta_{32} - 3\beta_{33}$$

$$C_5 = \frac{\alpha_1}{5!} + \frac{4\alpha_2}{15} + \frac{81\alpha_3}{40} - \tfrac{1}{4!}\beta_{11} - \frac{2}{3}\beta_{12} - \frac{27}{8}\beta_{13} - \tfrac{1}{3!}\beta_{21} - \frac{4}{3}\beta_{22} - \frac{9}{2}\beta_{23} - \frac{1}{2}\beta_{31} - 2\beta_{32} - \frac{9}{2}\beta_{33}$$

$$C_6 = \frac{\alpha_1}{6!} + \frac{4\alpha_2}{45} + \frac{81\alpha_3}{80} - \tfrac{1}{5!}\beta_{11} - \frac{4}{15}\beta_{12} - \frac{81}{40}\beta_{13} - \tfrac{1}{4!}\beta_{21} - \frac{2}{3}\beta_{22} - \frac{27}{8}\beta_{23} - \tfrac{1}{3!}\beta_{31} - \frac{4}{3}\beta_{32} - \frac{9}{2}\beta_{33}$$

$$C_7 = \frac{\alpha_1}{7!} + \frac{8\alpha_2}{315} + \frac{243\alpha_3}{560} - \tfrac{1}{6!}\beta_{11} - \frac{4}{45}\beta_{12} - \frac{81}{80}\beta_{13} - \tfrac{1}{5!}\beta_{21} - \frac{4}{15}\beta_{22} - \frac{81}{40}\beta_{23} - \tfrac{1}{4!}\beta_{31} - \frac{2}{3}\beta_{32} - \frac{27}{8}\beta_{33}$$

$$C_8 = \frac{\alpha_1}{8!} + \frac{2\alpha_2}{315} + \frac{729\alpha_3}{4480} - \tfrac{1}{7!}\beta_{11} - \frac{8}{315}\beta_{12} - \frac{243}{560}\beta_{13} - \tfrac{1}{6!}\beta_{21} - \frac{4}{45}\beta_{22} - \frac{81}{80}\beta_{23} - \tfrac{1}{5!}\beta_{31} - \frac{4}{15}\beta_{32} - \frac{81}{40}\beta_{33}$$

$$C_9 = \frac{\alpha_1}{9!} + \frac{4\alpha_2}{2835} + \frac{243\alpha_3}{4480} - \tfrac{1}{8!}\beta_{11} - \frac{2}{315}\beta_{12} - \frac{729}{4480}\beta_{13} - \tfrac{1}{7!}\beta_{21} - \frac{8}{315}\beta_{22} - \frac{243}{560}\beta_{23} - \tfrac{1}{6!}\beta_{31} - \frac{4}{45}\beta_{32} - \frac{81}{80}\beta_{33}$$

$$C_{10} = \frac{\alpha_1}{10!} + \frac{4\alpha_2}{14175} + \frac{729\alpha_3}{44800} - \tfrac{1}{9!}\beta_{11} - \frac{4}{2835}\beta_{12} - \frac{243}{4480}\beta_{13} - \tfrac{1}{8!}\beta_{21} - \frac{2}{315}\beta_{22} - \frac{729}{4480}\beta_{23}$$
$$- \tfrac{1}{7!}\beta_{31} - \frac{8}{315}\beta_{32} - \frac{243}{560}\beta_{33}$$

$$C_{11} = \frac{\alpha_1}{11!} + \frac{8\alpha_2}{155925} + 0.0044379\alpha_3 - \tfrac{1}{10!}\beta_{11} - \frac{4}{14175}\beta_{12} - \frac{729}{44800}\beta_{13} - \tfrac{1}{9!}\beta_{21} - \frac{4}{2835}\beta_{22} - \frac{243}{4480}\beta_{23}$$
$$- \tfrac{1}{8!}\beta_{31} - \frac{2}{315}\beta_{32} - \frac{729}{4480}\beta_{33}$$

$$C_{12} = \frac{\alpha_1}{12!} + \frac{4\alpha_2}{467775} + 0.0011099\alpha_3 - \tfrac{1}{11!}\beta_{11} - \frac{8}{155925}\beta_{12} - 0.0044379\beta_{13} - \tfrac{1}{10!}\beta_{21} - \frac{4}{14175}\beta_{22} - \frac{729}{44800}\beta_{23}$$
$$- \tfrac{1}{9!}\beta_{31} - \frac{4}{2835}\beta_{32} - \frac{243}{4480}\beta_{33}$$

$$C_{13} = \frac{\alpha_1}{13!} + \frac{8\alpha_2}{6081975} + 0.0002560\alpha_3 - \tfrac{1}{12!}\beta_{11} - \frac{4}{467775}\beta_{12} - 0.0011099\beta_{13} - \tfrac{1}{11!}\beta_{21} - \frac{8}{155925}\beta_{22} - 0.0044379\beta_{23}$$
$$- \tfrac{1}{10!}\beta_{31} - \frac{4}{14175}\beta_{32} - \frac{729}{44800}\beta_{33}$$

$$C_{14} = \frac{\alpha_1}{14!} + \frac{8\alpha_2}{42567525} + 0.0000549\alpha_3 - \tfrac{1}{13!}\beta_{11} - \frac{8\alpha_2}{6081975}\beta_{12} - 0.000256\beta_{13} - \tfrac{1}{12!}\beta_{21} - \frac{4}{467775}\beta_{22} - 0.0011099\beta_{2}$$
$$- \tfrac{1}{11!}\beta_{31} - \frac{8}{155925}\beta_{32} - 0.0044379\beta_{33}$$

(16)

imposing accuracy of order 14 on T_{n+3} to have
$C_0 = C_1 = C_2 = C_3 = \text{- - - -} = C_{14} = 0$ and $T_{n+3} = 0(h^{15})$,

solving gives;

α_0= -53.08, α_1=-139.212, α_2=191.292, β_{10} =19.2275, β_{11} = 139.5539, β_{12} =87.4317, β_{13} =0.15589, β_{20} =2.5703, β_{21} =15.2975, β_{22} =-15.2834, β_{23} =--0.0004, β_{30} =0.1279, β_{31} =4.8092, β_{32} =1.332 and β_{33} = --0.0005.

putting these values into equation (14) gives three-step third derivative method of the form

$$y_{n+3} = 53.08y_n + 139.21y_{n+1} - 191.29y_{n+2} + h\left(19.23y_n^{(1)} + 139.55y_{n+1}^{(1)} + 87.43y_{n+2}^{(1)} + 0.16y_{n+3}^{(1)}\right) + h^2\left(2.57y_n^{(11)} + 15.30y_{n+1}^{(11)} - 15.28y_{n+2}^{(11)}\right) + h^3\left(0.13y_n^{(111)} + 4.81y_n^{(111)} + 1.33y_n^{(111)}\right) \quad (17)$$

Derivation of the four - step fourth derivative scheme

Setting *k=4, l=4* in (3.1) gives

$$\alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_2 y_{n+2} + \alpha_3 y_{n+3} + \alpha_4 y_{n+4} = h\left(\beta_{10} y_n^{(1)} + \beta_{11} y_{n+1}^{(1)} + \beta_{12} y_{n+2}^{(1)} + \beta_{13} y_{n+3}^{(1)} + \beta_{14} y_{n+4}^{(1)}\right) + h^2\left(\beta_{20} y_n^{(11)} + \beta_{21} y_{n+1}^{(11)} + \beta_{22} y_{n+2}^{(11)} + \beta_{23} y_{n+3}^{(11)} + \beta_{24} y_{n+4}^{(11)}\right) + h^3\left(\beta_{30} y_n^{(111)} + \beta_{31} y_{n+1}^{(111)} + \beta_{32} y_{n+2}^{(111)} + \beta_{33} y_{n+3}^{(111)} + \beta_{34} y_{n+4}^{(111)}\right) + h^4\left(\beta_{40} y_n^{(1V)} + \beta_{41} y_{n+1}^{(1V)} + \beta_{42} y_{n+2}^{(1V)} + \beta_{43} y_{n+3}^{(1V)} + \beta_{44} y_{n+4}^{(1V)}\right) \quad (18)$$

with local truncation error

$$.T_{n+4} = \alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_2 y_{n+2} + \alpha_3 y_{n+3} + \alpha_3 y_{n+4} - h\left(\beta_{10} y_n^{(1)} + \beta_{11} y_{n+1}^{(1)} + \beta_{12} y_{n+2}^{(1)} + \beta_{13} y_{n+3}^{(1)} + \beta_{14} y_{n+4}^{(1)}\right) - h^2\left(\beta_{20} y_n^{(11)} + \beta_{21} y_{n+1}^{(11)} + \beta_{22} y_{n+2}^{(11)} + \beta_{23} y_{n+3}^{(11)} + \beta_{24} y_{n+4}^{(11)}\right) - -h^3\left(\beta_{30} y_n^{(111)} + \beta_{31} y_{n+1}^{(111)} + \beta_{32} y_{n+2}^{(111)} + \beta_{33} y_{n+3}^{(111)} + \beta_{34} y_{n+4}^{(111)}\right) - h^4\left(\beta_{40} y_n^{(1V)} + \beta_{41} y_{n+1}^{(1V)} + \beta_{42} y_{n+2}^{(1V)} + \beta_{43} y_{n+3}^{(1V)} + \beta_{44} y_{n+4}^{(1V)}\right) \quad (19)$$

adopting Taylor's series expansion of y_{n+1}, $y_{n+1}^{(1)}$, y_{n+2}, $y_{n+2}^{(1)}$, y_{n+3}, $y_{n+3}^{(1)}$, y_{n+4} $y_{n+4}^{(1)}$, $y_{n+1}^{(11)}$, $y_{n+2}^{(11)}$,, $y_{n+3}^{(11)}$, $y_{n+4}^{(11)}$, $y_{n+1}^{(111)}$, $y_{n+2}^{(111)}$,, $y_{n+3}^{(111)}$, $y_{n+4}^{(111)}$,. $y_{n+1}^{(1V)}$, $y_{n+2}^{(1V)}$, $y_{n+3}^{(1V)}$ *and* $y_{n+4}^{(1V)}$
.in (19) and combining terms in equal powers of h gives

$$T_{n+4} = C_0 y_n + C_1 h y_n^{(1)} + +C_2 h^2 y_n^{(11)} + +C_3 h^3 y_n^{(111)} + \ldots\ldots + +C_n h^n y_n^{(n)}$$

where

$$C_0 = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$$

$$C_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 - \beta_{10} - \beta_{11} - \beta_{12} - \beta_{13} - \beta_{14}$$

$$C_2 = \frac{\alpha_1}{2} + 2\alpha_2 + \frac{9\alpha_3}{2} + 8\alpha_4 - \beta_{11} - 2\beta_{12} - 3\beta_{13} - 4\beta_{14} - \beta_{20} - \beta_{21} - \beta_{22} - \beta_{23} - \beta_{24}$$

$$C_3 = \frac{\alpha_1}{3!} + \frac{4\alpha_2}{3} + \frac{9\alpha_3}{2} + \frac{32\alpha_4}{3} - \tfrac{1}{2}\beta_{11} - 2\beta_{12} - \frac{9}{2}\beta_{13} - 8\beta_{14} - \beta_{21} - 2\beta_{22} - 3\beta_{23} - 4\beta_{24}$$
$$-\beta_{30} - \beta_{31} - \beta_{32} - \beta_{33} - \beta_{34}$$

$$C_4 = \frac{\alpha_1}{4!} + \frac{2\alpha_2}{3} + \frac{27\alpha_3}{8} + \frac{32\alpha_4}{3} - \tfrac{1}{3!}\beta_{11} - \frac{4}{3}\beta_{12} - \frac{9}{2}\beta_{13} + \frac{32\beta_{14}}{3} - \tfrac{1}{2}\beta_{21} - 2\beta_{22} - \frac{9}{2}\beta_{23} - 8\beta_{24}$$
$$-\beta_{31} - 2\beta_{32} - 3\beta_{33} - 4\beta_{34} - \beta_{40} - \beta_{41} - \beta_{42} - \beta_{43} - \beta_{44}$$

$$C_5 = \frac{\alpha_1}{5!} + \frac{4\alpha_2}{15} + \frac{81\alpha_3}{40} + \frac{128\alpha_4}{15} - \tfrac{1}{4!}\beta_{11} - \frac{2}{3}\beta_{12} - \frac{27}{8}\beta_{13} - \frac{32\beta_{14}}{3} - \tfrac{1}{3!}\beta_{21} - \frac{4}{3}\beta_{22} - \frac{9}{2}\beta_{23} + \frac{32\beta_{24}}{3}$$
$$-\tfrac{1}{2}\beta_{31} - 2\beta_{32} - \frac{9}{2}\beta_{33} - 8\beta_{34} - \beta_{41} - 2\beta_{42} - 3\beta_{43} - 4\beta_{44}$$

$$C_6 = \frac{\alpha_1}{6!} + \frac{4\alpha_2}{45} + \frac{81\alpha_3}{80} + \frac{256\alpha_4}{45} - \tfrac{1}{5!}\beta_{11} - \frac{4}{15}\beta_{12} - \frac{81}{40}\beta_{13} - \frac{128\beta_{14}}{15} - \tfrac{1}{4!}\beta_{21} - \frac{2}{3}\beta_{22} - \frac{27}{8}\beta_{23} - \frac{32\beta_{24}}{3}$$
$$-\tfrac{1}{3!}\beta_{31} - \frac{4}{3}\beta_{32} - \frac{9}{2}\beta_{33} - \frac{32\beta_{34}}{3} - \beta_{41} - 2\beta_{42} - 3\beta_{43} - 4\beta_{44}$$

$$C_7 = \frac{\alpha_1}{7!} + \frac{8\alpha_2}{315} + \frac{243\alpha_3}{560} + \frac{1024\alpha_4}{315} - \tfrac{1}{6!}\beta_{11} - \frac{4}{45}\beta_{12} - \frac{81}{80}\beta_{13} \frac{128\beta_{14}}{15} - \tfrac{1}{5!}\beta_{21} - \frac{4}{15}\beta_{22} - \frac{81}{40}\beta_{23} - \frac{128\beta_{24}}{15}$$
$$-\tfrac{1}{4!}\beta_{31} - \frac{2}{3}\beta_{32} - \frac{27}{8}\beta_{33} - \frac{32\beta_{34}}{3} - \tfrac{1}{3!}\beta_{41} - \frac{4}{3}\beta_{42} - \frac{9}{2}\beta_{43} - \frac{32\beta_{44}}{3}$$

$$C_8 = \frac{\alpha_1}{8!} + \frac{2\alpha_2}{315} + \frac{729\alpha_3}{4480} + \frac{512\alpha_4}{315} - \tfrac{1}{7!}\beta_{11} - \frac{8}{315}\beta_{12} - \frac{243}{560}\beta_{13} - \frac{1024\beta_{14}}{315} - \tfrac{1}{6!}\beta_{21} - \frac{4}{45}\beta_{22} - \frac{81}{80}\beta_{23}$$
$$-\frac{128\beta_{24}}{15} - \tfrac{1}{5!}\beta_{31} - \frac{4}{15}\beta_{32} - \frac{81}{40}\beta_{33} - \frac{128\beta_{34}}{15} - \tfrac{1}{4!}\beta_{41} - \frac{2}{3}\beta_{42} - \frac{27}{8}\beta_{43} - \frac{32\beta_{44}}{3}$$

$$C_9 = \frac{\alpha_1}{9!} + \frac{4\alpha_2}{2835} + \frac{243\alpha_3}{4480} + \frac{2048\alpha_4}{2835} - \tfrac{1}{8!}\beta_{11} - \frac{2}{315}\beta_{12} - \frac{729}{4480}\beta_{13} - \frac{512\beta_{14}}{315} - \tfrac{1}{7!}\beta_{21} - \frac{8}{315}\beta_{22} - \frac{243}{560}\beta_{23} - \frac{1024\beta_{24}}{315}$$
$$-\tfrac{1}{6!}\beta_{31} - \frac{4}{45}\beta_{32} - \frac{81}{80}\beta_{33} - \frac{128\beta_{34}}{15} - \tfrac{1}{5!}\beta_{41} - \frac{4}{15}\beta_{42} - \frac{81}{40}\beta_{43} - \frac{128\beta_{44}}{15}$$

$$C_{10} = \frac{\alpha_1}{10!} + \frac{4\alpha_2}{14175} + \frac{729\alpha_3}{44800} + \frac{4096\alpha_4}{14175} - \tfrac{1}{9!}\beta_{11} - \frac{4}{2835}\beta_{12} - \frac{243\beta_{133}}{4480} - \frac{2048\beta_{14}}{2835} - \tfrac{1}{8!}\beta_{21} - \frac{2}{315}\beta_{22} - \frac{729}{4480}\beta_{23} - \frac{512}{3}$$
$$-\tfrac{1}{7!}\beta_{31} - \frac{8}{315}\beta_{32} - \frac{243}{560}\beta_{33} - \frac{1024\beta_{34}}{315} - \tfrac{1}{6!}\beta_{41} - \frac{4}{45}\beta_{42} - \frac{81}{80}\beta_{43} - \frac{128\beta_{44}}{15}$$

$$C_{11} = \frac{\alpha_1}{11!} + \frac{8\alpha_2}{155925} + 0.004438\alpha_3 + 0.1050762\alpha_4 - \frac{1}{10!}\beta_{11} - \frac{4}{14175}\beta_{12} - \frac{729}{44800}\beta_{13} - \frac{4096}{14175}\beta_{14} - \tfrac{1}{9!}\beta_{21} - \frac{4}{2835}\beta_{22}$$
$$-\frac{243\beta_{23}}{4480} - \frac{2048\beta_{24}}{2835} - \tfrac{1}{8!}\beta_{31} - \frac{2}{315}\beta_{32} - \frac{729}{4480}\beta_{33} - \frac{512\beta_{34}}{315} - \tfrac{1}{7!}\beta_{41} - \frac{8}{315}\beta_{42} - \frac{243}{560}\beta_{43} - \frac{1024\beta_{44}}{315}$$

$$C_9=\frac{\alpha_1}{9!}+\frac{4\alpha_2}{2835}+\frac{243\alpha_3}{4480}+\frac{2048\alpha_4}{2835}-\tfrac{1}{8!}\beta_{11}-\frac{2}{315}\beta_{12}-\frac{729}{4480}\beta_{13}-\frac{512\beta_{14}}{315}-\tfrac{1}{7!}\beta_{21}-\frac{8}{315}\beta_{22}-\frac{243}{560}\beta_{23}-\frac{1024\beta_{24}}{315}$$
$$-\tfrac{1}{6!}\beta_{31}-\frac{4}{45}\beta_{32}-\frac{81}{80}\beta_{33}-\frac{128\beta_{34}}{15}-\tfrac{1}{5!}\beta_{41}-\frac{4}{15}\beta_{42}-\frac{81}{40}\beta_{43}-\frac{128\beta_{44}}{15}$$

$$C_{10}=\frac{\alpha_1}{10!}+\frac{4\alpha_2}{14175}+\frac{729\alpha_3}{44800}+\frac{4096\alpha_4}{14175}-\tfrac{1}{9!}\beta_{11}-\frac{4}{2835}\beta_{12}-\frac{243\beta_{133}}{4480}-\frac{2048\beta_{14}}{2835}-\tfrac{1}{8!}\beta_{21}-\frac{2}{315}\beta_{22}-\frac{729}{4480}\beta_{23}-\frac{512\beta_{24}}{315}$$
$$-\tfrac{1}{7!}\beta_{31}-\frac{8}{315}\beta_{32}-\frac{243}{560}\beta_{33}-\frac{1024\beta_{34}}{315}-\tfrac{1}{6!}\beta_{41}-\frac{4}{45}\beta_{42}-\frac{81}{80}\beta_{43}-\frac{128\beta_{44}}{15}$$

$$C_{11}=\frac{\alpha_1}{11!}+\frac{8\alpha_2}{155925}+0.004438\alpha_3+0.1050762\alpha_4-\frac{1}{10!}\beta_{11}-\frac{4}{14175}\beta_{12}-\frac{729}{44800}\beta_{13}-\frac{4096}{14175}\beta_{14}-\tfrac{1}{9!}\beta_{21}-\frac{4}{2835}\beta_{22}$$
$$-\frac{243\beta_{23}}{4480}-\frac{2048\beta_{24}}{2835}-\tfrac{1}{8!}\beta_{31}-\frac{2}{315}\beta_{32}-\frac{729}{4480}\beta_{33}-\frac{512\beta_{34}}{315}-\tfrac{1}{7!}\beta_{41}-\frac{8}{315}\beta_{42}-\frac{243}{560}\beta_{43}-\frac{1024\beta_{44}}{315}$$

$$C_{12}=\frac{\alpha_1}{12!}+\frac{4\alpha_2}{467775}+0.00411095\alpha_3+0.035025\alpha_4-\frac{1}{11!}\beta_{11}-\frac{8}{155925}\beta_{12}-0.004438\beta_{13}-0.1050762\beta_{14}-\frac{1}{10!}\beta_{21}$$
$$-\frac{4}{14175}\beta_{22}-\frac{729}{44800}\beta_{23}-\frac{4096}{14175}\beta_{24}-\tfrac{1}{9!}\beta_{31}-\frac{4}{2835}\beta_{32}-\frac{243\beta_{33}}{4480}-\frac{2048\beta_{34}}{2835}-\tfrac{1}{8!}\beta_{41}-\frac{2}{315}\beta_{42}-\frac{729}{4480}\beta_{43}-\frac{512\beta_{44}}{315}$$

$$C_{13}=\frac{\alpha_1}{13!}+\frac{8\alpha_2}{86081075}+0.000256\alpha_3+0.010777\alpha_4-\frac{1}{12!}\beta_{11}-\frac{4}{467775}\beta_{12}-0.00411095\beta_{13}$$
$$-0.035025\beta_{14}-\frac{1}{11!}\beta_{21}-\frac{8}{155925}\beta_{22}-0.004438\beta_{23}-0.1050762\beta_{24}-\frac{1}{10!}\beta_{31}-\frac{4}{14175}\beta_{32}-\frac{729}{44800}\beta_{33}-\frac{4096}{14175}\beta_{34}$$
$$-\tfrac{1}{9!}\beta_{41}-\frac{4}{2835}\beta_{42}-\frac{243\beta_{43}}{4480}-\frac{2048\beta_{44}}{2835}$$

$$C_{14}=\frac{\alpha_1}{14!}+\frac{8\alpha_2}{42567525}+0.0000549\alpha_3+0.0030792\alpha_4-\frac{1}{13!}\beta_{11}-\frac{8}{86081075}\beta_{12}-0.000256\beta_{13}-0.010777\beta_{14}$$
$$-\frac{1}{12!}\beta_{21}-\frac{4}{467775}\beta_{22}-0.00411095\beta_{23}-0.035025\beta_{24}-\frac{1}{11!}\beta_{31}-\frac{8}{155925}\beta_{32}-0.004438\beta_{33}-0.1050762\beta_{34}$$
$$-\frac{1}{10!}\beta_{41}-\frac{4}{14175}\beta_{42}-\frac{729}{44800}\beta_{43}-\frac{4096}{14175}\beta_{44}$$

$$C_{15}=\frac{\alpha_1}{15!}+2.5*10^{-8}\alpha_2+0.000011\alpha_3+0.00821\alpha_4-\frac{1}{14!}\beta_{11}-\frac{8\beta_{12}}{42567525}-0.0000549\beta_{13}-0.0030792\beta_{14}-\frac{1}{13!}\beta_{21}$$
$$-\frac{8}{86081075}\beta_{22}-0.000256\beta_{23}-0.010777\beta_{24}-\frac{1}{12!}\beta_{31}-\frac{4}{467775}\beta_{32}-0.00411095\beta_{33}-0.035025\beta_{34}$$
$$-\frac{1}{11!}\beta_{41}-\frac{8}{155925}\beta_{42}-0.004438\beta_{43}-0.1050762\beta_{44}$$

$$C_{16}=\frac{\alpha_1}{16!}+3.0*10^{-9}\alpha_2+0.0000021\alpha_3+0.0002053\alpha_4-\frac{1}{15!}\beta_{11}+2.5*10^{-8}\beta_{12}-0.000011\beta_{13}-0.00821\beta_{14}$$
$$-\frac{1}{14!}\beta_{21}-\frac{8\beta_{22}}{42567525}-0.0000549\beta_{23}-0.0030792\beta_{24}-\frac{1}{13!}\beta_{31}-\frac{8}{86081075}\beta_{32}-0.000256\beta_{33}-0.010777\beta_{34}$$
$$-\frac{1}{12!}\beta_{41}-\frac{4}{467775}\beta_{42}-0.00411095\beta_{43}-0.035025\beta_{44}$$

$$C_{15}=\frac{\alpha_1}{15!}+2.5*10^{-8}\alpha_2+0.000011\alpha_3+0.00821\alpha_4-\frac{1}{14!}\beta_{11}-\frac{8\beta_{12}}{42567525}-0.0000549\beta_{13}-0.0030792\beta_{14}-\frac{1}{13!}\beta_{21}$$
$$-\frac{8}{86081075}\beta_{22}-0.000256\beta_{23}-0.010777\beta_{24}-\frac{1}{12!}\beta_{31}-\frac{4}{467775}\beta_{32}-0.00411095\beta_{33}-0.035025\beta_{34}$$
$$-\frac{1}{11!}\beta_{41}-\frac{8}{155925}\beta_{42}-0.004438\beta_{43}-0.1050762\beta_{44}$$

$$C_{16}=\frac{\alpha_1}{16!}+3.0*10^{-9}\alpha_2+0.0000021\alpha_3+0.0002053\alpha_4-\frac{1}{15!}\beta_{11}+2.5*10^{-8}\beta_{12}-0.000011\beta_{13}-0.00821\beta_{14}$$
$$-\frac{1}{14!}\beta_{21}-\frac{8\beta_{22}}{42567525}-0.0000549\beta_{23}-0.0030792\beta_{24}-\frac{1}{13!}\beta_{31}-\frac{8}{86081075}\beta_{32}-0.000256\beta_{33}-0.010777\beta_{34}$$
$$-\frac{1}{12!}\beta_{41}-\frac{4}{467775}\beta_{42}-0.00411095\beta_{43}-0.035025\beta_{44}$$

$$C_{17}=\frac{\alpha_1}{17!}+3.7*10^{-10}\alpha_2+0.0000004\alpha_3+0.0000483\alpha_4-\frac{1}{16!}\beta_{11}-3.0*10^{-9}\beta_{12}-0.0000021\beta_{13}-0.0002053\beta_{14}$$
$$-\frac{1}{15!}\beta_{21}+2.5*10^{-8}\beta_{22}-0.000011\beta_{23}-0.008211\beta_{24}-\frac{1}{14!}\beta_{31}-\frac{8\beta_{32}}{42567525}-0.0000549\beta_{33}-0.0030792\beta_{34}$$
$$-\frac{1}{13!}\beta_{41}-\frac{8}{86081075}\beta_{42}-0.000256\beta_{43}-0.010777\beta_{44}$$

$$C_{18}=\frac{\alpha_1}{18!}+4.1*10^{-11}\alpha_2+6.1*10^{-8}\alpha_3+0.0000107\alpha_4-\frac{1}{17!}\beta_{11}-3.7*10^{-10}\beta_{12}-0.0000004\beta_{13}-0.0000483\beta_{14}$$
$$-\frac{1}{16!}\beta_{21}-3.0*10^{-9}\beta_{22}-0.0000021\beta_{23}-0.0002053\beta_{24}-\frac{1}{15!}\beta_{31}-2.5*10^{-8}\beta_{32}-0.000011\beta_{33}-0.008211\beta_{34}$$
$$-\frac{1}{14!}\beta_{41}-\frac{8\beta_{42}}{42567525}-0.0000549\beta_{43}-0.0030792\beta_{44}$$

$$C_{19}=\frac{\alpha_1}{19!}+4.3*10^{-12}\alpha_2+9*10^{-9}\alpha_3+0.0000023\alpha_4-\frac{1}{18!}\beta_{11}-4.1*10^{-11}\beta_{12}-6*10^{-11}\beta_{13}-0.0000107\beta_{14}$$
$$-\frac{1}{17!}\beta_{21}-3.7*10^{-10}\beta_{22}-0.0000004\beta_{23}-0.0000483\beta_{24}-\frac{1}{16!}\beta_{31}-3*10^{-9}\beta_{32}-0.000002\beta_{33}-0.000205\beta_{34}$$
$$-\frac{1}{15!}\beta_{41}-2.5*10^{-8}\beta_{42}-0.000011\beta_{43}-0.008211\beta_{44}$$

$$C_{20}=\frac{\alpha_1}{20!}+4.3*10^{-13}\alpha_2+10^{-9}\alpha_3+0.0000005\alpha_4-\frac{1}{19!}\beta_{11}-4.3*10^{-12}\beta_{12}-9*10^{-9}\beta_{13}+0.0000023\beta_{14}-\frac{1}{18!}\beta_{21}$$
$$-4.1*10^{-11}\beta_{22}-6*10^{-11}\beta_{23}-0.0000107\beta_{24}-\frac{1}{17!}\beta_{31}-3.7*10^{-10}\beta_{32}-0.0000004\beta_{33}-0.0000483\beta_{34}-\frac{1}{16!}\beta_{41}$$
$$-3*10^{-9}\beta_{42}-0.000002\beta_{43}-0.000205\beta_{44}$$

$$C_{21}=\frac{\alpha_1}{21!}+4.1*10^{-14}\alpha_2+2*10^{-10}\alpha_3+8.6*10^{-10}\alpha_4-\frac{1}{20!}\beta_{11}+4.3*10^{-13}\beta_{12}+10^{-9}\beta_{133}+0.0000005\beta_{14}-\frac{1}{19!}\beta_{21}$$
$$-4.3*10^{-12}\beta_{22}-9*10^{-9}\beta_{23}+0.0000023\beta_{24}-\frac{1}{18!}\beta_{31}-4.1*10^{-11}\beta_{32}-6*10^{-11}\beta_{33}-0.0000107\beta_{34}$$
$$-\frac{1}{17!}\beta_{41}-3.7*10^{-10}\beta_{42}-0.0000004\beta_{43}-0.0000483\beta_{44}$$

$$C_{22}=\frac{\alpha_1}{22!}+3.7*10^{-15}\alpha_2+2.8*10^{-11}\alpha_3+1.5*10^{-8}\alpha_4-\frac{1}{21!}\beta_{11}-4.1*10^{-14}\beta_{12}-2*10^{-10}\beta_{13}$$
$$-8.6*10^{-10}\beta_{14}-\frac{1}{20!}\beta_{21}-4.3*10^{-13}\beta_{22}-10^{-9}\beta_{23}-0.0000005\beta_{24}-\frac{1}{19!}\beta_{31}$$
$$-4.3*10^{-12}\beta_{32}-9*10^{-9}\beta_{33}+0.0000023\beta_{34}-\frac{1}{18!}\beta_{41}-4.1*10^{-11}\beta_{42}-6*10^{-11}\beta_{43}-0.0000107\beta_{44}$$

$$-4.3*10^{-12}\beta_{22}-9*10^{-9}\beta_{23}+0.0000023\beta_{24}-\frac{1}{18!}\beta_{31}-4.1*10^{-11}\beta_{32}-6*10^{-11}\beta_{33}-0.0000107\beta_{34}$$

$$-\frac{1}{17!}\beta_{41}-3.7*10^{-10}\beta_{42}-0.0000004\beta_{43}-0.0000483\beta_{44}$$

$$C_{22}=\frac{\alpha_1}{22!}+3.7*10^{-15}\alpha_2+2.8*10^{-11}\alpha_3+1.5*10^{-8}\alpha_4-\frac{1}{21!}\beta_{11}-4.1*10^{-14}\beta_{12}-2*10^{-10}\beta_{13}$$

$$-8.6*10^{-10}\beta_{14}-\frac{1}{20!}\beta_{21}-4.3*10^{-13}\beta_{22}-10^{-9}\beta_{23}-0.0000005\beta_{24}-\frac{1}{19!}\beta_{31}$$

$$-4.3*10^{-12}\beta_{32}-9*10^{-9}\beta_{33}+0.0000023\beta_{34}-\frac{1}{18!}\beta_{41}-4.1*10^{-11}\beta_{42}-6*10^{-11}\beta_{43}-0.0000107\beta_{44}$$

$$C_{23}=\frac{\alpha_1}{23!}+3.2*10^{-16}\alpha_2+3.6*10^{-12}\alpha_3+2*10^{-9}\alpha_4-\frac{1}{22!}\beta_{11}-3.7*10^{-15}\beta_{12}-2.8*10^{-11}\beta_{13}-1.5*10^{-8}\beta_{14}$$

$$-\frac{1}{21!}\beta_{21}-4.1*10^{-14}\beta_{22}-2*10^{-10}\beta_{23}-8.6*10^{-10}\beta_{24}-\frac{1}{20!}\beta_{31}-4.3*10^{-13}\beta_{32}$$

$$-10^{-9}\beta_{33}-0.0000005\beta_{34}-\frac{1}{19!}\beta_{41}-4.3*10^{-12}\beta_{42}-9*10^{-9}\beta_{43}+0.0000023\beta_{44}$$

(20)

solving gives
α_0 = -7.5992, α_1= 7.0581, α_2= 0.5411, α_3= -0.0001, β_{10} = 3.1378, β_{11} =4.8746, β_{12} = 0.1286, β_{13} = -0.0006, β_{14} = 0, β_{20} = 0.5192, β_{21} = -1.0680, β_{22} = 0.0303, β_{23} = 0, β_{24} = -0.07661, β_{30} = 0.0387, β_{31} = 0.1886, β_{32} = -0.0136, β_{33} = 0, β_{34} = 0, β_{41} = 0.0165, β_{42} = -0.0018, β_{43} = 0, β_{44} = 0.
putting these values into equation (18) gives four – step, fourth derivative method of the form:

$$y_{n+4}=-0.54y_{n+2}-7.60y_{n+1}+7.60y_n+h\left(0.13y^{(1)}_{n+2}+4.87y^{(1)}_{n+1}+3.14y^{(1)}_{n}\right)$$
$$+h^2\left(0.03y^{(11)}_{n+2}-1.07y^{(11)}_{n+1}+0.52y^{(11)}_{n}\right)+h^3\left(-0.01y^{(111)}_{n+2}+0.19y^{(111)}_{n+1}+0.04y^{(111)}_{n}\right)-h^4\left(0.02y^{(1v)}_{n+1}\right)$$
(21)

3. Results and Analysis

Implementation:
The generated schemes were used to evaluate the test equations:
(i) $y^1=x+y, y(0)=1,\ x\epsilon[0,1]\ with\ h=0.1$
(ii) $y^1=-10(y-x^3+3x^2),\ y(0)=1,\ x\epsilon[0,1]\ with\ h=0.1$

According to [5], accuracy of a numerical method can be improved by increasing k of the method. This was investigated by increasing k at constant *l* [6]. It was established that there is optimal accuracy after which accuracy started to decrease as k increased; the work confirmed that the optimal accuracy for a first derivative multistep method was obtained when k was set at 2.
Also, [7] reported that accuracy of a numerical method can be improved by increasing *l*. This was studied by increasing *l* at constant k [10]. It was also established that there is optimal accuracy after which accuracy was decreasing as *l* was increasing. The work showed that optimal accuracy for the one - step method was obtained at *l*=3.
In this study, both parameters k and *l* were increased simultaneously to determine its effect on the accuracy of multiderivative multistep method. Tables 1 - 4 showed that the accuracy increased from k=1;*l*=1 to k=2:*l*=2 and started to decrease from k=2:*l*=2 to k=3:*l*=3 and also decreased from k=3:*l*=3 to k=4:*l*=4 as the k and *l* were increased together, showing optimal accuracy at k=2: *l*=2.

TABLE 1: Result of Increasing Derivative and Step Number Simultaneously for Problem 1.

X	Exact Solution	k=1; *l*=1	k=2; *l*=2	k=3; *l*=3	k=4; *l*=4
1.2	4.44023384547309	4.44593789553446	4.44385168589713	4.36937240789451	-0.2465294173321
2.4	18.6463527612832	18.6842452556505	18.6703826120990	18.1783269168011	-0.0333296816617
3.6	68.5964688873560	68.7852612194024	68.7161743628944	66.2780327278759	-0.1303871702343
4.8	237.220835037470	238.056944255861	237.750894322222	227.012096264151	-0.1615012301942
6.0	799.857586985471	803.329053459631	802.058009922223	757.714680710864	-0.2066311951432
7.2	2670.66152878884	2684.49828211996	2679.43071160517	2503.64776168687	-0.2487821582903

8.4	8884.73349539973	8938.35277719081	8918.70991718344	8241.23168922763	-0.2915662915211
9.6	29518.9631311546	29722.5046309030	29647.9190625770	27090.1360834544	-0.3342158480106
10.8	98029.8022727638	98790.3834092049	98511.6009375571	89005.6139301460	-0.3768940080299

TABLE 2: Result of Increasing Derivative and Step Number Simultaneously for Problem 2.

X	Exact Solution	k=1; l=1	k=2; *l*=2	k=3; *l* =3	k=4; *l*=4
1.2	1.72800614421235	1.72933339285850	1.72874484982808	1.72836753058262	-0.2814338449108
2.4	13.8240000000378	13.8253333333334	13.8255692212801	13.8247404039684	-2.9239162887048
3.6	46.6560000000001	46.6573333333334	46.6583989400371	46.6571181853344	-10.723718947458
4.8	110.592000000000	110.593333333333	110.595228658823	110.593495966732	-26.430224535619
6.0	215.999999999999	216.001333333333	216.004058377608	216.001873748127	-52.803402183607
7.2	373.247999999999	373.249333333332	373.252888096394	373.250251529537	-92.603113969221
8.4	592.703999999997	592.705333333330	592.709717815180	592.706629310865	-148.58922305280
9.6	884.735999999995	884.737333333328	884.742547533964	884.739007092283	-223.52159258373
10.8	1259.71199999999	1259.71333333333	1259.71937725275	1259.71538487377	-320.16008571153

TABLE 3: Table of Errors for Problem 1

X	Exact solution	k=1; *l*=1	k=2; *l*=2	k=3; *l*=3	k=4; *l*=4
1.2	4.4402338454730943	7.341662e-01	6.612877e-03	7.086144e-02	4.686763e+00
2.4	18.646352761283204	2.672636e+00	4.393292e-02	4.680258e-01	1.867968e+01
3.6	68.596468887355954	9.661582e+00	2.189026e-01	2.318436e+00	6.872686e+01
4.8	237.22083503746970	3.471935e+01	9.695256e-01	1.020874e+01	2.373823e+02
6.0	799.85758698547022	1.241275e+02	4.025677e+00	4.214291e+01	8.000642e+02
7.2	2670.6615287888362	4.418005e+02	1.604686e+01	1.670138e+02	2.670910e+03
8.4	8884.7334953997160	1.566326e+03	6.218799e+01	6.435018e+02	8.885025e+03
9.6	29518.963131154538	5.533896e+03	2.360849e+02	2.428827e+03	2.951930e+04
10.8	98029.802272763322	1.949111e+04	8.822478e+02	9.024188e+03	9.803018e+04

TABLE 4: Table of Errors for Problem 2

X	Exact solution	k=1; *l*=1	k=2; *l*=2	k=3; *l*=3	k=4;l=4
1.2	1.7280061442123531	1.333333e-03	7.387056e-04	3.613864e-04	2.009440e+00
2.4	13.824000000037765	1.333333e-03	1.569221e-03	7.404039e-04	1.674792e+01
3.6	46.656000000000077	1.333333e-03	2.398940e-03	1.118185e-03	5.737972e+01
4.8	110.59199999999993	1.333333e-03	3.228659e-03	1.495967e-03	1.370222e+02
6.0	215.99999999999943	1.333333e-03	4.058378e-03	1.873748e-03	2.688034e+02
7.2	373.24799999999851	1.333333e-03	4.888096e-03	2.251530e-03	4.658511e+02
8.4	592.70399999999711	1.333333e-03	5.717815e-03	2.629311e-03	7.412932e+02
9.6	884.73599999999499	1.333333e-03	6.547534e-03	3.007092e-03	1.108258e+03
10.8	1259.7119999999999	1.333333e-03	7.377253e-03	3.384874e-03	1.579872e+03

4. Conclusion

In this study, the accuracy pattern of increasing k and *l* together in an implicit multiderivative linear multistep method of the form in equation (1) showed that better accuracy was achieved at k=2; *l*=2, accuracy reduced from k=3; *l* =3 to k=4; *l* =4, suggesting that two step second derivative scheme (k=2; *l*=2) has the optimal accuracy when k and *l* were increased simultaneously.

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